Robust Coupling Linear State Observer-Controller Design for MIMO Systems with Mismatched and Unstructured Uncertainties

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Abstract- In paper [1] a new robust decoupling state observer including an extra non-linear term has been proposed by Gu and Poon. This extra term is used to overcome the difficulty due to the unstructured parameter perturbation. It was highlighted that the nonlinear observer scheme requires the solution to a quadratic matrix inequality which is not straightforward. This inequality was rewritten as a quadratic Riccati equation by introducing a parameter $\varepsilon$ which was successfully solved by proposed algorithms. However, the proposed observer possesses some shortages, for example, strictly speaking, the observer error dynamics is a function of error, plant state and extra term, then the origin $e=0$ is not equilibrium point because plant state and extra term can increase unboundedly. Therefore, selected Lyapunov function is not a good candidate and observer state error does not go to zero.

In this paper, a simple linear Luenberger robust coupling state observer—controller design for the linear MIMO systems with mismatched and unstructured parameter perturbations is considered. The same observer problem mentioned above is solved only by using a simple linear Luenberger scheme without including any extra non-linear term. The simple linear Luenberger observer scheme is considered as a completely dual form to the linear multivariable controller and investigated together with controller. Then the design parameters of coupling closed-loop observer-controller system are selected such that the observer error dynamics and plant state equations are globally asymptotically stable. The stability conditions are formulated in terms of two quadratic Riccati equations and matrix inequality. Therefore, the coupling observer–controller laws can be constructed from the positive definite solutions of these algebraic Riccati equations and matrix inequality. Reduced design methodology of completely symmetric dual coupling closed loop observer-controller system is also presented.

Keywords- State observer, coupling system, robustness, asymptotic stability, parameter perturbation.

I. INTRODUCTION

In paper [1], a new robust decoupling state observer scheme including an extra non-linear term has been proposed. The purpose of a proposed non-linear state observer is to estimate the unavailable state variables of the linear MIMO systems with unstructured parameter perturbations. The main results of this paper which determines the construction of a Luenberger observer combined with extra non-linear term are given in the following theorem [1].

**Theorem 1:** Given a linear uncertain MIMO system with unstructured parameter perturbation

$$\dot{x}(t) = \left[A + \Delta A(\sigma)\right]x(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where $x(t)$ is a state n-vector, $u(t)$ is a control input m-vector, $y(t)$ is a output p-vector and $\|\Delta A(\sigma)\|_2 < \delta$, $\delta$ is a constant, then the robust non-linear state observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + B\left[y(t) - \hat{y}(t)\right]$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t)$$

where $\hat{y}(t)$ is the residual defined as follows

$$r(t) = y(t) - \hat{y}(t) = C[x(t) - \hat{x}(t)] = Ce(t)$$

estimates the unavailable state variables $x(t)$, or the observer state error equation:

$$\dot{\hat{x}}(t) = \Delta A(\sigma)\hat{x}(t) + H\hat{e}(t)$$

$$\hat{e}(t) = \Delta A(\sigma)e(t)$$

converges to zero, if the following extra non-linear term using to overcome the difficulty due to unstructured parameter perturbations $\Delta A(\sigma)$ is satisfied:

$$\alpha(\hat{x}(t), r(t)) = \frac{\delta^2 x^T(t)\hat{x}(t)}{2r^T(t)r(t)}$$

$$V(e(t)) = \frac{1}{2}e^T(t)Pe(t)$$

satisfying the following quadratic matrix inequality:
\[(A-HC)^T P + P (A-HC) + 2P^2 + \delta^2 I < 0 \]  \hspace{1cm} (7)

where \(H\) is such that \(A-HC\) is stable. It was highlighted that the nonlinear observer scheme requires the solution to a quadratic matrix inequality which is not straightforward. This inequality was rewritten as a quadratic Riccati equation by introducing a parameter \(\varepsilon\) which was successfully solved by proposed algorithms. For using design procedure of [2] to solving ARE in [1] has been assumed that an unique positive definite solution \(P\) of quadratic Riccati equation exists if and only if the system matrix

\[
\begin{bmatrix}
A_0 - HC & \sqrt{2} \delta I \\
\delta I & 0
\end{bmatrix}
\]

is observable and controllable; \(H^\infty\) norm of this matrix transfer function is less than or equal to \(\gamma \) where \(\gamma = 1/\sqrt{2}\).

However, the proposed observer possesses some shortages:

1) Observer (2) involves an extra non-linear term (5) depending on estimated state variables, residual and their non-linear functions. Practical implementation of this non-linear term is difficult.

2) Strictly speaking, the observer state variable \(\hat{x}(t)\) cannot be considered as the estimate of plant state variable \(x(t)\) in (1) since as seen from (4) observer state error dynamics is function of \(e(t)\), \(x(t)\) and \(\alpha(t)\), then origin \(e = 0\) is not equilibrium point for (4), because of plant state variables \(x(t)\), if they are unstable and extra term \(\alpha(t)\) can increase unboundedly as pointed in [1] also.

Consequently, \(e(t)\) does not go to zero as \(t \to \infty\). For this reason we can conclude that the selected Lyapunov function (6) is not a good candidate. Therefore, the proposed non-linear robust state observer [1] needs to be improved. Moreover, an observer for uncertain systems should be designed together with the state controller because observer error equation (4) depends of \(x(t)\), too.

It should be noted that various coupling observer/controller design for linear multivariable systems without parameter perturbations and closed-loop stability analysis are considered by several authors and presented for example in [3]. Recently, a new observer and coupling observer based controller are designed in the behavioral context by [4]. The robust observer-based control of linear systems with mismatched but structured uncertainties via LMI approach is designed by [5].

In this paper, a simple linear Luenberger robust state observer design for linear MIMO systems with mismatched and unstructured system matrix parameter perturbations is considered. The same decoupling observer problem mentioned in [1] is solved only by using a simple linear Luenberger scheme without including any extra non-linear term. The simple linear Luenberger scheme is considered as a completely dual form to the linear multivariable controller based on dual time-invariant observer concept [6] and investigated together with controller. Then the design parameters of coupling closed-loop observer controller system are selected such that the observer error dynamics and plant state equations are globally asymptotically stable. The stability conditions are formulated in terms of two quadratic Riccati equations and matrix inequality, which can be successfully solved by using very well known algorithms [1], [2]. Reduced design methodology of completely symmetric dual coupling closed loop observer-controller system is presented also.

II. MAIN RESULTS

Consider a linear uncertain MIMO system with parameter perturbation described by (1). Let us form a simple linear Luenberger observer as follows:

\[
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - Bv(t)
\]

\[
\dot{y}(t) = C\hat{x}(t) + Du(t)
\]

where \(v(t)\) is the linear Luenberger compensative term which can be selected as follows:

\[
v(t) = -L(y(t) - \dot{\hat{y}}(t)) = -Lr(t)
\]

\[
= -L Ce(t) = -k_{OBS} B^T Re(t)
\]

where \(L\) is the observer gain \((p \times p)\) matrix in term of \(r(t)\), \(k_{OBS}\) is a constant gain parameter, \(R\) is the positive-definite matrix to be selected. Here we assume that there exist a matrix \(R\) such that the structural constraint similar to [7], satisfying \(L C = k_{OBS} B^T R = K_{OBS}\), where \(K_{OBS}\) is the observer gain \((p \times n)\) matrix in term of \(r(t)\). Note that this coupling observer scheme uses only the residual as an input signal.

The dual linear control law is defined as:

\[
u(t) = -G\hat{x}(t) = -k_{CON} B^T P\hat{x}(t)
\]

where \(G = k_{CON} B^T P, G\) the control gain matrix, \(k_{CON}\) is a constant gain parameter and \(P\) is a positive definite matrix to be selected.

Then the observer error dynamics can be obtained as

\[
\dot{e}(t) = \dot{\hat{x}}(t) - \hat{x}(t) = Ae(t) + A(A\sigma)x(t) + Bv(t)
\]

\[
= [A - k_{OBS} BB^T R] e(t) + A(A\sigma)x(t)
\]

and closed-loop system state equation (1) becomes

\[
\dot{x}(t) = Ax(t) + A(A\sigma)x(t) - k_{CON} BB^T P\hat{x}(t)
\]

\[
= Ax(t) + A(A\sigma)x(t) - k_{CON} BB^T \hat{P} x(t) + e(t)
\]

\[
= [A - k_{CON} BB^T P] x(t) + A(A\sigma)x(t) + \hat{P} e(t)
\]

(12)
Thus, the dual coupling closed-loop observer/control system equations are presented as follows

\[
\begin{bmatrix}
\Delta x(t) \\
\Delta \xi(t)
\end{bmatrix} =
\begin{bmatrix}
-A - k_{\text{con}} B B^T R \\
k_{\text{con}} B B^T A - k_{\text{con}} B B^T P + \Delta A
\end{bmatrix}
\begin{bmatrix}
e(t) \\
x(t)
\end{bmatrix}
\] (13)

We assume that the pairs (A,B) and (C,A) are completely controllable and observable respectively. This implies that we can find the design parameters such that all the eigenvalues of the matrices \( A - k_{\text{con}} B B^T P \) and \( A - k_{\text{obs}} B B^T R \) have a desired location in the left-half of s-plane. Moreover, we need some additional system structure conditions for solving algebraic Riccati equations.

The main results which determine the design parameters of the coupling observer-control configuration are given in the following theorem.

**Theorem 2:** Given a linear uncertain MIMO system (1) with control law (10) and simple linear Luenberger observer (8) with the dual compensative term (9), then the dual coupling closed-loop observer-control error system (13) with mismatched and unstructured parameter perturbations is globally asymptotically stable, if the following conditions are satisfied:

\[
P A + A^T P + 2 \max_{\sigma} \Delta A^T (\sigma) \Delta A(\sigma) + P^2
\]

\[
-2 k_{\text{con}} P B B^T P = -Q_{\text{con}}
\]

\[
R A + A^T R + R^2 - 2 k_{\text{obs}} R B B^T R = -Q_{\text{obs}}
\]

where \( Q_{\text{con}} \) and \( Q_{\text{obs}} \) are positive definite matrices and matrix inequality

\[
H = \begin{bmatrix}
Q_{\text{obs}} & -k_{\text{con}} P B B^T P \\
-k_{\text{con}} P B B^T P & Q_{\text{con}}
\end{bmatrix} > 0
\]

or the Schur complement

\[
H = Q_{\text{obs}} - k_{\text{con}}^2 P B B^T P Q_{\text{con}}^{-1} P B B^T P > 0
\]

**Proof:** To examine the stability of dual coupling closed-loop observer-control system (13), we define a Lyapunov V-function candidate as a full quadratic form of \( e(t) \) and \( x(t) \):

\[
V(x(t), e(t)) = \begin{bmatrix}
e(t) \\
x(t)
\end{bmatrix}^T
\begin{bmatrix}
R & 0 \\
0 & P
\end{bmatrix}
\begin{bmatrix}
e(t) \\
x(t)
\end{bmatrix}
\]

\[
= x^T(t) P x(t) + e^T(t) R e(t)
\]

where \( P \) and \( R \) are positive-definite matrices. The time-derivative of (17) along the closed-loop trajectories of (13) is given by:

\[
\dot{V}(x(t), e(t)) = 2 x^T(t) P (A - k_{\text{con}} B B^T P) x(t) + 2 x^T(t) P A \Delta A(\sigma) x(t) + 2 k_{\text{con}} x^T(t) P B B^T P e(t) + 2 e^T(t) R (A - k_{\text{obs}} B B^T R) e(t) + 2 e^T(t) R \Delta A(\sigma) x(t)
\]

Since

\[
2 x^T(t) P A \Delta A(\sigma) x(t) \leq x^T(t) P P^T x(t)
\]

\[
+ x^T(t) \Delta A^T(\sigma) \Delta A(\sigma) x(t)
\]

\[
2 e^T(t) R \Delta A(\sigma) x(t) \leq e^T(t) R R^T e(t)
\]

\[
x^T(t) \Delta A^T(\sigma) \Delta A(\sigma) x(t)
\]

Then

\[
\dot{V}(x(t), e(t)) \leq x^T(t) (P A + A^T P) x(t) - 2 k_{\text{con}} x^T(t) P B B^T P x(t) + x^T(t) \Delta A^T(\sigma) \Delta A(\sigma) x(t) + 2 k_{\text{con}} x^T(t) P B B^T P e(t) + e^T(t) (R A + A^T R) e(t) - 2 k_{\text{obs}} e^T(t) R B B^T R e(t) + e^T(t) R R^T e(t) + x^T(t) \Delta A^T(\sigma) \Delta A(\sigma) x(t)
\]

\[
= x^T(t) P A + A^T P + 2 \max_{\sigma} \Delta A^T(\sigma) \Delta A(\sigma) + P^2
\]

\[
+ P^2 - 2 k_{\text{con}} P B B^T P x(t) + 2 k_{\text{con}} x^T(t) P B B^T P e(t) + e^T(t) (R A + A^T R) e(t) + x^T(t) \Delta A^T(\sigma) \Delta A(\sigma) x(t)
\]

Letting

\[
PA + A^T P + 2 \max_{\sigma} \Delta A^T(\sigma) \Delta A(\sigma) + P^2
\]

\[
- 2 k_{\text{con}} P B B^T P = -Q_{\text{con}}
\]

\[
R A + A^T R + R^2 - 2 k_{\text{obs}} R B B^T R = -Q_{\text{obs}}
\]

Then (21) leads to

\[
\dot{V}(x(t), e(t)) \leq - e^T(t) H e(t)
\]

\[
= \begin{bmatrix}
e(t) \\
x(t)
\end{bmatrix}^T \begin{bmatrix}
Q_{\text{obs}} & -k_{\text{con}} P B B^T P \\
-k_{\text{con}} P B B^T P & Q_{\text{con}}
\end{bmatrix} \begin{bmatrix}
e(t) \\
x(t)
\end{bmatrix}
\]

In view of (24) if the conditions of (14), (15) and (16) are satisfied, then (24) reduces to \( \dot{V}(x(t), e(t)) < 0 \). Therefore, the coupling closed-loop observer-control system with parameter perturbation is globally asymptotically stable, that is \( x(t) \) and \( e(t) \) converge to zero. Theorem 2 is proved.
**Solution algorithm to the quadratic Riccati equations:**
The algebraic Riccati equations (22), (23) which are completely identical to [1], [2], [8], and matrix inequality can be solved by using very well known methods [1], [2] as follows:

Step 1: Choose any positive definite matrices $P$, $R$ and gain constants $k_{\text{CON}}, k_{\text{OBS}}$ such that $A - K_{\text{CON}} BB^T P$ and $A - K_{\text{OBS}} BB^T R$ are stable. Initialize $\epsilon$ to some starting value, e.g. $\epsilon = 1$.

Step 2: Determine whether the transformed similar to [1], [2] Riccati equations (22), (23) have (for $Q_{\text{CON}}, Q_{\text{OBS}}$) some positive definite solutions for given $Q_{\text{CON}}$ and $Q_{\text{OBS}}$. If positive definite solutions exist, the algorithm succeeds.

Step 3: Determine whether the already linear matrix inequality (16) for determined $P, R, Q_{\text{CON}}$ and $Q_{\text{OBS}}$ has a positive definite solution. If the solution exists go to Step 4. Otherwise, stop and the algorithm fails.

Step 4: $\epsilon = \epsilon/2$ if $\epsilon$ is less than some computational accuracy $\epsilon_0$ then stop, the algorithm fails. Otherwise go to Step 3.

Step 5: the algorithm effectively succeeds and use (9), (10) to compute observer and controller gain matrices $K_{\text{OBS}}$ and $G$.

**III. REDUCED DESIGN**

Letting in (17) $P=R$ and in (9), (10) $k_{\text{OBS}} = k_{\text{CON}} = k$ , that is considered coupling observer-controller system is completely symmetric and dual form, then the stability conditions (14), (15) and (16) are reduced to:

\[
PA + A^T P + 2 \max_{\sigma} \Delta A^T(\sigma) \Delta A(\sigma) + P^2 \\
- 2kPBB^T P = -Q, \quad Q = Q^T > 0
\]

where $Q = Q_{\text{CON}}$ then $Q_{\text{OBS}} = Q + 2 \max_{\sigma} \Delta A^T(\sigma) A(\sigma)$

\[
H = \begin{bmatrix}
Q + 2 \max_{\sigma} \Delta A^T(\sigma) A(\sigma) & -kPBB^T P \\
-kPBB^T P & Q
\end{bmatrix} > 0
\]  
(26a)

or the Schur complement

\[
H = Q + 2 \max_{\sigma} \Delta A^T(\sigma) \Delta A(\sigma) \\
- k^2 PBB^T PQ^{-1} PBB^T P > 0
\]  
(26b)

Therefore, the design procedure is simplified considerably.

**IV. CONCLUSION**

We have considered the simple linear Luenberger robust coupling state observer for the linear MIMO systems with mismatched and unstructured parameter perturbations. Observer scheme is considered as a completely dual form to the linear multivariable controller and investigated together with controller. Then the design parameters of coupling closed-loop observer-controller system are selected such that the observer error dynamics and plant state equations are globally asymptotically stable. The stability conditions are formulated in terms of two quadratic Riccati equations and one matrix inequality, which can be solved by using effective algorithms [1], [2]. Reduced design of symmetric dual coupling closed loop observer-controller system is presented also.

**REFERENCES**


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